

On the Structure of Sets with Few Three-Term Arithmetic Progressions

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1 Introduction

Given a function $f : \mathbb{F}_{p^n} \rightarrow [0, 1]$, and a subset $W \subseteq \mathbb{F}_{p^n}$, we define

$$\mathbb{E}(f|W) = |W|^{-1} \sum_{m \in W} f(m).$$

If no set W is given, then we just assume $W = \mathbb{F}_{p^n}$, and then we get

$$\mathbb{E}(f) = \mathbb{E}(f|\mathbb{F}_{p^n}) = p^{-n} \sum_{m \in \mathbb{F}_{p^n}} f(m).$$

Define

$$\Lambda_3(f) = p^{-2n} \sum_{m,d} f(m)f(m+d)f(m+2d).$$

In the case where f is an indicator function for some set $S \subseteq \mathbb{F}_{p^n}$, we have that $\Lambda_3(f)$ is the normalized count of the number of three-term arithmetic progressions $m, m+d, m+2d \in S$. Note that $\Lambda_3(f) \geq 0$, unless $\mathbb{E}(f) = 0$, because of the contribution of trivial progressions where $d = 0$.

Of central importance to the subject of additive combinatorics is that of determining when a subset of the integers $\{1, \dots, N\}$ contains a k -term arithmetic progression. This subject has a long history, and we will not mention it here; however, the specific problem in this area which motivated our paper, and which is due to B. Green [1], is as follows:

Problem. Given $0 < \alpha \leq 1$, suppose $S \subseteq \mathbb{F}_p$ satisfies $|S| \geq \alpha p$, and has the least number of three-term arithmetic progressions. What is $\Lambda_3(S)$?

It seems that the only hope of answering a question like this is to understand the structure of these sets S . In this paper we address the analogous problem in \mathbb{F}_{p^n} , where p and α are held fixed, while n tends to infinity. The results we prove are not of a type that would allow us to deduce $\Lambda_3(S)$, but they do reveal that these sets S are very highly structured. Such results can perhaps be deduced from the work of B. Green [2], which makes use of the Szemerédi regularity lemma, but our theorems below are proved using basic harmonic analysis.

Theorem 1 *Let $0 < \alpha \leq 1$. Suppose that S is a subset of \mathbb{F}_{p^n} , such that $\Lambda_3(S)$ is minimal, subject to the constraint*

$$|S| \geq \alpha p^n.$$

Then, there exists a subgroup (or subspace)

$$W \leq \mathbb{F}_{p^n}, \dim(W) = n - o(n),$$

such that S is approximately a union of $p^{o(n)}$ cosets of W ; more precisely, there is a set A of size $p^{o(n)}$ such that

$$|S \Delta A + W| = o(p^n).^1$$

Our second theorem is a slightly more abstract version of this one, where instead of sets S , we have a function $f : \mathbb{F}_{p^n} \rightarrow [0, 1]$.

Theorem 2 *Let $0 < \alpha \leq 1$. Suppose that*

$$f : \mathbb{F}_{p^n} \rightarrow [0, 1]$$

such that $\Lambda_3(f)$ is minimal, subject to the constraint that

$$\mathbb{E}(f) \geq \alpha > 0.$$

Then, there exists a subgroup $W \subseteq \mathbb{F}_{p^n}$ of dimension $n - o(n)$, such that f is approximately an indicator function on cosets of W , in the following sense: There is a function

$$h : \mathbb{F}_{p^n} \rightarrow \{0, 1\},$$

which is constant on cosets of W (which means $h(a) = h(a + w)$ for all $w \in W$), such that

$$\mathbb{E}(|f(m) - h(m)|) = o(1).$$

¹The notation $B \Delta C$ means the symmetric difference between B and C .

It would seem that Theorem 1 is a corollary of Theorem 2; however, with a little thought one sees this is not the case. Nonetheless, we will prove a third theorem, from which we will deduce both Theorem 1 and Theorem 2.

2 Proofs

2.1 Additional Notation

We will require a little more notation.

Given any three subsets $U, V, W \subseteq \mathbb{F}_{p^n}$, define

$$T_3(f|U, V, W) = \sum_{m \in U, m+d \in V, m+2d \in W} f(m)f(m+d)f(m+2d).$$

We note that this implies $T_3(1|U, U, U)$ is the number of three-term progressions belonging to a set U .

Given a subspace W of \mathbb{F}_{p^n} , and given a function

$$f : \mathbb{F}_{p^n} \rightarrow [0, 1],$$

we define

$$f_W(m) = \frac{1}{|W|}(f * W)(m) = \frac{1}{|W|} \sum_{w \in W} f(m+w).$$

This function has a number of properties: First, we note that $f_W(m)$ is constant on cosets of W , in the sense that

$$\text{for all } w \in W, f_W(m) = f_W(m+w).$$

Thus, it makes sense to write

$$f_W(m+W) = f_W(m).$$

We also have that

$$\mathbb{E}(f_W) = \mathbb{E}(f). \tag{1}$$

Finally, if V is the orthogonal complement of W (with respect to the standard basis), then

$$\text{if } v \in V, \text{ then } \hat{f}_W(v) = \hat{f}(v); \text{ and, if } v \notin V, \text{ then } \hat{f}_W(v) = 0. \tag{2}$$

We will also define the L^2 norm of a function $f : \mathbb{F}_{p^n} \rightarrow \mathbb{C}$ to be

$$\|f\|_2 = \left(p^{-n} \sum_m |f(m)|^2 \right)^{1/2}.$$

2.2 Theorem 3, and Proofs of Theorems 1 and 2

Theorems 1 and 2 are corollaries of the following theorem:

Theorem 3 *Let $\epsilon > 0$, and suppose that*

$$f : \mathbb{F}_{p^n} \rightarrow [0, 1]$$

has the following property: For every subspace W of \mathbb{F}_{p^n} of codimension at most Δ^{-2} , where

$$\Delta = (\epsilon^6/2^{13}p^2) \exp(-16\epsilon^{-1}c_p \log p),$$

where c_p is a certain constant appearing in Theorem 4 below, suppose that

$$\mathbb{E}(|f(m) - f_W(m)|) > \epsilon.$$

Then, there exists a function

$$g : \mathbb{F}_{p^n} \rightarrow [0, 1]$$

such that

$$\mathbb{E}(g) = \mathbb{E}(f), \text{ and } \Lambda_3(g) < \Lambda_3(f) - \Delta.$$

Comment. Using the Lemma 1 below we can deduce the stronger conclusion that there exists

$$g : \mathbb{F}_{p^n} \rightarrow \{0, 1\}$$

(so, g is an indicator function) such that

$$\mathbb{E}(g) \geq \mathbb{E}(f), \text{ and } \Lambda_3(g) < \Lambda_3(f) - \Delta + O(p^{-n/3}). \quad (3)$$

Lemma 1 *Suppose that $j : \mathbb{F}_{p^n} \rightarrow [0, 1]$. There exists an indicator function $j_2 : \mathbb{F}_{p^n} \rightarrow \{0, 1\}$, such that*

$$\mathbb{E}(j_2) \geq \mathbb{E}(j), \Lambda_3(j_2) = \Lambda_3(j) + O(p^{-n/3}),$$

and such that for every subspace W of codimension at most $n^{1/2}$ we have² that for every $m \in \mathbb{F}_{p^n}$,

$$(j_2)_W(m) = j_W(m) + O(1/n).$$

²The codimension $n^{1/2}$ condition can be improved; however, it is good enough for our purposes, and it is larger than Δ^{-2} , where $\epsilon = 1/\log \log n$, as will appear in later applications.

In order to prove this lemma we will need to use a theorem of Hoeffding (see [3] or [4, Theorem 5.7])

Proposition 1 *Suppose that z_1, \dots, z_r are independent real random variables with $|z_i| \leq 1$. Let $\mu = \mathbb{E}(z_1 + \dots + z_r)$, and let $\Sigma = z_1 + \dots + z_r$. Then,*

$$\mathbb{P}(|\Sigma - \mu| > rt) \leq 2 \exp(-rt^2/2).$$

Proof of the Lemma. The proof of this lemma is standard: Given j as in the theorem above, let j_0 be a random function from \mathbb{F}_{p^n} to $\{0, 1\}$, where $j_0(m) = 1$ with probability $j(m)$, and equals 0 with probability $1 - j(m)$; moreover, $j_0(m)$ is independent of all the other $j_0(m')$. Then, one can easily show that with probability $1 - o(1)$,

$$p^{-n} \sum_m j_0(m) = \mathbb{E}(j) + O(p^{-n/3}), \text{ and } \Lambda_3(j_0) = \Lambda_3(j) + O(p^{-n/3}). \quad (4)$$

Furthermore, we claim that with probability $1 - o(1)$ we will have that for any subspace W of codimension at most $n^{1/2}$,

$$(j_0)_W(m) = j_W(m) + O(1/n). \quad (5)$$

This can be seen as follows: For a fixed W we need an upper bound on the probability that

$$|(j_0)_W(m) - j_W(m)| > 1/n.$$

This is the same as showing

$$|\Sigma| > |W|/n,$$

where

$$\Sigma = \sum_{w \in W} z_w(m), \text{ where } z_w(m) = j_0(m+w) - j(m+w).$$

Note that all the z_w are independent and satisfy $|z_w| \leq 1$ and $\mathbb{E}(z_w) = 0$. So, from Proposition 1 we deduce that

$$\mathbb{P}(|\Sigma| > |W|/n) \leq 2 \exp(-|W|/2n^2).$$

Now, since the number of such subspaces W is at most the number of sequences of $n^{1/2}$ possible basis vectors, which is $O(p^{n^{3/2}})$, we deduce that

the probability that there exists a subspace W of codimension at most $n^{1/2}$ satisfying

$$|(j_0)_W(m) - j_W(m)| > 1/n$$

is $O(p^{n^{3/2}} \exp(-|W|/2n^2)) = o(1)$. Thus, (5) holds for all such W with probability $1 - o(1)$ (in fact, the explicit constant in the $O(1)$ can be taken to be 1 once n is sufficiently large).

We deduce now that there is an instantiation of j_0 , call it j_1 , such that both (4) and (5) hold. Then, by reassigning at most $O(p^{2n/3})$ places m where $j_1(m) = 0$ to the value 1, or from the value 0 to the value 1, we arrive at a function j_2 having the claimed properties of the lemma. ■

Proof of Theorem 1. To prove Theorem 1, we begin by letting f be the indicator function for the set S , and we let

$$\epsilon = \frac{1}{\log \log n}.$$

Now suppose that

$$\mathbb{E}(|f(m) - f_W(m)|) \leq \epsilon, \tag{6}$$

for some subspace W of codimension at most Δ^{-2} . Let $h(m)$ be $f_W(m)$ rounded to the nearest integer. Clearly, $h(m)$ is constant on cosets of W , and from the fact that

$$|h(m) - f_W(m)| \leq |f(m) - f_W(m)|,$$

we deduce that

$$\begin{aligned} \mathbb{E}(|f(m) - h(m)|) &\leq \mathbb{E}(|h(m) - f_W(m)|) + \mathbb{E}(|f(m) - f_W(m)|) \\ &\leq 2\mathbb{E}(|f(m) - f_W(m)|) \\ &\leq 2\epsilon. \end{aligned}$$

But since h is constant on cosets of W , and only assumes the values 0 or 1, we deduce that h is the indicator function for some set of the form $A + W$. Thus, we deduce

$$|S \Delta A + W| \leq 2\epsilon p^n,$$

where W has dimension $n - o(n)$. This then proves Theorem 1 under the assumption (6).

Next, suppose that

$$\mathbb{E}(|f(m) - f_W(m)|) > \epsilon. \quad (7)$$

for every subspace W of codimension at most Δ^{-2} . Then, from the comment following Theorem 3, there exists an indicator function g satisfying (3). If we let S' be the set for which g is an indicator function, then one sees that S' has fewer three-term arithmetic progressions than does S , while $\mathbb{E}(S') \geq \mathbb{E}(S)$. This is a contradiction, and thus the theorem is proved. ■

Proof of Theorem 2. Let $j(m) = f(m)$, and then let

$$\ell(m) = j_2(m) : \mathbb{F}_{p^n} \rightarrow \{0, 1\},$$

where $j_2(m)$ is as given in Lemma 1. Note that this implies that

$$\mathbb{E}(\ell) \geq \mathbb{E}(f), \quad \Lambda_3(\ell) = \Lambda_3(f) + O(p^{-n/3}),$$

and that for any subspace W of codimension at most $n^{1/2}$,

$$\ell_W(m) = f_W(m) + O(1/n). \quad (8)$$

Next let

$$\epsilon = \frac{1}{\log \log n},$$

and suppose that there exists a subspace W of codimension at most Δ^{-2} such that

$$\mathbb{E}(|\ell(m) - \ell_W(m)|) \leq \epsilon. \quad (9)$$

Then, if we let $h(m)$ equal $f_W(m)$ rounded to the nearest integer, we will have from (8) that

$$\begin{aligned} \mathbb{E}(|h(m) - f_W(m)|) &\leq \mathbb{E}(|\ell(m) - f_W(m)|) \\ &\leq \mathbb{E}(|\ell(m) - \ell_W(m)|) + O(1/n) \\ &\leq \epsilon + O(1/n). \end{aligned} \quad (10)$$

Let V be the orthogonal complement of W . From (10) we know that at most

$$(\epsilon^{1/2} + O(\epsilon^{-1/2}/n))|V|$$

values $v \in V$ satisfy

$$|h(v) - f_W(v)| \geq \epsilon^{1/2}.$$

Let $V' \subseteq V$ be those $v \in V$ satisfying the reverse inequality

$$|h(v) - f_W(v)| < \epsilon^{1/2}.$$

Suppose $v \in V'$ and $h(v) = 0$. Then, $f_W(v) < \epsilon^{1/2}$, and we have

$$\sum_{m \in v+W} |f(m) - h(m)| = |W|f_W(v) < |W|\epsilon^{1/2}. \quad (11)$$

On the other hand, if $v \in V'$ and $h(v) = 1$, then $f_W(v) > 1 - \epsilon^{1/2}$, and so

$$\sum_{m \in v+W} |f(m) - h(m)| = |W|(1 - f_W(v)) < |W|\epsilon^{1/2}. \quad (12)$$

Combining (11) with (12) we deduce that

$$\begin{aligned} \mathbb{E}(|f(m) - h(m)|) &\leq \epsilon^{1/2} + (|V| - |V'|)|V|^{-1} \\ &\leq 2\epsilon^{1/2} + O(\epsilon^{-1/2}/n). \end{aligned} \quad (13)$$

Our theorem is now proved in this case (assuming there exists a subspace W satisfying (9)).

To complete the proof, we will assume that there are no subspaces of codimension at most Δ^{-2} satisfying (9). Since ℓ then satisfies the hypotheses of Theorem 3, we deduce from Theorem 3 that there exists a function $g : \mathbb{F}_{p^n} \rightarrow [0, 1]$ such that

$$\mathbb{E}(g) = \mathbb{E}(\ell) \geq \mathbb{E}(f) \geq \alpha,$$

and

$$\Lambda_3(g) < \Lambda_3(\ell) - \Delta = \Lambda_3(f) - \Delta + O(p^{-n/3}).$$

This then contradicts the fact that $\Lambda_3(f)$ was minimal, given $\mathbb{E}(f) \geq \alpha$. Our theorem is now proved. \blacksquare

3 Proof of Theorem 3

Let Δ be as in the statement of Theorem 3.

As is well-known,

$$\Lambda_3(f) = p^{-3n} \sum_{a \in \mathbb{F}_{p^n}} \hat{f}(a)^2 \hat{f}(-2a).$$

If we let A denote the set of all $a \in \mathbb{F}_{p^n}$ where

$$|\hat{f}(a)| > \Delta p^n,$$

then we clearly have

$$\Lambda_3(f) = p^{-3n} \sum_{a \in A} \hat{f}(a)^2 \hat{f}(-2a) + E, \quad (14)$$

where

$$|E| \leq \Delta p^{-n} \|\hat{f}\|_2^2 \leq \Delta. \quad (15)$$

A simple application of Parseval's identity also shows that $|A|$ is small: We have

$$|A| \Delta^2 p^{2n} \leq p^n \|\hat{f}\|_2^2 \leq p^{2n},$$

which implies

$$|A| \leq \Delta^{-2}.$$

Let V be the additive subgroup of \mathbb{F}_{p^n} generated by the elements of A , and let W be the orthogonal complement of V ; that is,

$$W = \{w \in \mathbb{F}_{p^n} : \text{for every } v \in V, w \cdot v = 0\}^3.$$

From (14), (15), and (2) we deduce that

$$\Lambda_3(f_W) \leq \Lambda_3(f) + \Delta. \quad (16)$$

Since W is an additive subgroup of \mathbb{F}_{p^n} , we will use the standard representation for the cosets of W , given by

$$v + W, \text{ where } v \in V.$$

This canonical representation for the cosets of W has the following important property.

³The product $w \cdot v$ here denotes the dot product with respect to the standard basis of the vector space \mathbb{F}_{p^n} , not the product defined for the multiplicative structure of \mathbb{F}_{p^n} .

Lemma 2 Suppose that $h : \mathbb{F}_{p^n} \rightarrow [0, 1]$. Then,

$$T_3(h) = \sum_{\substack{v_1, v_2, v_3 \in V \\ v_1 + v_3 = 2v_2}} T_3(h|_{v_1 + W, v_2 + W, v_3 + W}).$$

Proof. The lemma will follow if we can just show that $v_1 + w_1, v_2 + w_2, v_3 + w_3$, $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$, are in arithmetic progression implies v_1, v_2, v_3 are in arithmetic progression: If

$$(v_1 + w_1) + (v_3 + w_3) = 2(v_2 + w_2),$$

then

$$v_1 + v_3 - 2v_2 = -w_1 - w_3 + 2w_2.$$

Now, as $V \cap W = \{0\}$, we deduce that

$$v_1 + v_3 - 2v_2 = 0,$$

whence v_1, v_2, v_3 are in arithmetic progression. ■

Now let

$$V' := \{v \in V : f_W(v + W) \in [\epsilon/4, 1 - \epsilon/4]\}; \quad (17)$$

that is, these cosets are all the places where f_W is not “too close” to being an indicator function.

3.1 Construction of the Function g

To construct the function g with the properties claimed by our Theorem, we start with the following lemma:

Lemma 3 Suppose $h_1 : \mathbb{F}_{p^n} \rightarrow [0, 1]$, let $\beta = \mathbb{E}(h_1)$, and let $h_2(n) = 1 - h_1(n)$. Then,

$$\Lambda_3(h_1) + \Lambda_3(h_2) = 1 - 3\beta + 3\beta^2.$$

Proof. We first realize that for $a \neq 0$, $\hat{h}_1(a) = -\hat{h}_2(a)$. Thus,

$$\begin{aligned} \Lambda_3(h_1) + \Lambda_3(h_2) &= p^{-3n} \sum_a (\hat{h}_1(a)^2 \hat{h}_1(-2a) + \hat{h}_2(a)^2 \hat{h}_2(-2a)) \\ &= p^{-3n} (\hat{h}_1(0)^3 + \hat{h}_2(0)^3) \\ &= \beta^3 + (1 - \beta)^3. \quad \blacksquare \end{aligned}$$

Now, let ℓ be the unique integer satisfying

$$4/\epsilon \leq p^\ell < 4p/\epsilon,$$

and let S be any subspace of W of codimension ℓ . Let T be the complement of S relative to W (not *orthogonal* complement, as we have used earlier), and set

$$\beta = \frac{|T|}{|W|} = \frac{|W| - |S|}{|W|} = 1 - p^{-\ell} \geq 1 - \epsilon/4,$$

which is the density of T relative to W . Then, from the above lemma, we deduce that

$$T_3(S) + T_3(T) = (1 - 3\beta + 3\beta^2)|W|^2,$$

$T_3(S)$ clearly equals $(1 - \beta)^2|W|^2$, because given any pair of elements $m, m + d \in S$, since S is a subspace we also must have $m + 2d \in S$; and, note that there are $(1 - \beta)^2|W|^2$ ordered pairs $m, m + d$ in S . Thus, we deduce

$$T_3(T) = (2\beta^2 - \beta)|W|^2.$$

We also have that if $b_1 + W, b_2 + W, b_3 + W$ are cosets that are in arithmetic progression, in the sense that there is a triple $m, m + d, m + 2d$, belonging to $b_1 + W, b_2 + W$, and $b_3 + W$, respectively, then

$$T_3(1|b_1 + T, b_2 + T, b_3 + T) = (2\beta^2 - \beta)|W|^2.$$

We now define the function $g : \mathbb{F}_{p^n} \rightarrow [0, 1]$ as follows: Given $v \in V, w \in W$, we have

$$g(v + w) = \begin{cases} f_W(v), & \text{if } v \notin V'; \\ \beta^{-1}T(w)f_W(v), & \text{if } v \in V'. \end{cases}$$

It is easy to see that

$$\mathbb{E}(g) = \mathbb{E}(f_W) = \mathbb{E}(f);$$

We also observe, from Lemma 2, that

$$T_3(g) = \sum_{\substack{v_1, v_2, v_3 \in V \\ v_1 + v_3 = 2v_2}} T_3(g|v_1 + W, v_2 + W, v_3 + W).$$

This sum has eight types of terms, according to whether each of v_1, v_2, v_3 lie in V' or not.

First, consider the case where all of

$$v_1, v_2, v_3 \in V'. \quad (18)$$

In this case we have

$$\begin{aligned} T_3(g|v_1 + W, v_2 + W, v_3 + W) &= \beta^{-3} f_W(v_1) f_W(v_2) f_W(v_3) T_3(T) \\ &= f_W(v_1) f_W(v_2) f_W(v_3) |W|^2 (2\beta^{-1} - \beta^{-2}) \\ &\leq f_W(v_1) f_W(v_2) f_W(v_3) |W|^2 (1 - p^{-2\ell}) \\ &< f_W(v_1) f_W(v_2) f_W(v_3) |W|^2 (1 - \epsilon^2/16p^2). \end{aligned}$$

This last inequality follows from the fact that

$$p^\ell < 4p/\epsilon.$$

Now, as

$$T_3(f_W|v_1 + W, v_2 + W, v_3 + W) = f_W(v_1) f_W(v_2) f_W(v_3) |W|^2,$$

we deduce that if (18) holds, then

$$T_3(g|v_1 + W, v_2 + W, v_3 + W) \leq T_3(f_W|v_1 + W, v_2 + W, v_3 + W) (1 - \epsilon^2/16p^2).$$

On the other hand, if any of v_1, v_2, v_3 fail to lie in V' , then we will get that

$$T_3(g|v_1 + W, v_2 + W, v_3 + W) = T_3(f_W|v_1 + W, v_2 + W, v_3 + W).$$

To see this, consider all the cases where v_1 fails to lie in V' . In this case, we clearly have

$$\begin{aligned} T_3(g|v_1 + W, v_2 + W, v_3 + W) &= \sum_{m_1 \in v_2 + W, m_2 \in v_3 + W} f_W(v_1) g(m_1) g(m_2) \\ &= f_W(v_1) (|W|^2 f_W(v_2) f_W(v_3)) \\ &= T_3(f_W|v_1 + W, v_2 + W, v_3 + W). \end{aligned}$$

The cases where v_2 or v_3 fail to lie in V' are identical to this one.

Putting together the above observations we deduce that

$$\begin{aligned} T_3(g) &\leq T_3(f_W) - (\epsilon^2/16p^2) \sum_{\substack{v_1, v_2, v_3 \in V' \\ v_1 + v_3 = 2v_2}} T_3(f_W|v_1 + W, v_2 + W, v_3 + W) \\ &\leq T_3(f_W) - (\epsilon^5/1024p^2) |W|^2 T_3(V'). \end{aligned} \quad (19)$$

This last inequality follows from the fact that $f_W(v) \geq \epsilon/4$ for $v \in V'$.

3.2 A Lower Bound for $|V'|$

In order to give a lower bound for $T_3(V')$, we will first need a lower bound for $|V'|$.

We begin by noting that if v belongs to V , but not V' , then either $f_W(v) < \epsilon/4$ or $f_W(v) > 1 - \epsilon/4$. Suppose the former holds. Then, we have

$$\begin{aligned} \sum_{m \in v+W} |f(m) - f_W(m)| &\leq |W|f_W(v) + \sum_{m \in v+W} f(m) = 2|W|f_W(v) \\ &< \epsilon|W|/2. \end{aligned} \quad (20)$$

On the other hand, if $f_W(v) > 1 - \epsilon/4$, then we have

$$\begin{aligned} \sum_{m \in v+W} |f(m) - f_W(m)| &\leq \sum_{m \in v+V} (1 - f(m)) + \sum_{m \in v+W} (1 - f_W(m)) \\ &= 2|W| - 2|W|f_W(v) \\ &< \epsilon|W|/2. \end{aligned} \quad (21)$$

Putting together (20) and (21) we deduce that

$$\sum_{v \in V \setminus V'} \sum_{m \in v+W} |f(m) - f_W(m)| < \epsilon|W|(|V| - |V'|)/2.$$

We also have the trivial upper bound

$$\sum_{v \in V'} \sum_{m \in v+W} |f(m) - f_W(m)| \leq |W||V'|.$$

Thus,

$$|V|^{-1}(|V'| + \epsilon(|V| - |V'|)/2) > \mathbb{E}(|f(m) - f_W(m)|) > \epsilon.$$

(The second inequality is one of the hypotheses of the Theorem.) It follows that

$$|V'| > \frac{\epsilon|V|}{2(1 - \epsilon/2)} > \epsilon|V|/2. \quad (22)$$

3.3 Some Results of Meshulam and Varnavides

Using our lower bound for $|V'|$, we will need the following result of Meshulam [5] to obtain a lower bound for $T_3(V')$:

Theorem 4 *Suppose that $S \subseteq \mathbb{F}_{p^n}$ satisfies $|S| \geq c_p p^n/n$, where $c_p > 0$ is a certain constant depending only on p . Then, S contains a non-trivial three-term arithmetic progression.*

If we combine this with an idea of Varnavides [6], we get the following theorem.

Theorem 5 *Suppose that $S \subseteq \mathbb{F}_{p^n}$ satisfies $|S| = \alpha p^n$. Then,*

$$\Lambda_3(S) \geq (\alpha/2) \exp(-8\alpha^{-1}c_p \log p).$$

Proof of the Theorem. From Meshulam's theorem we know that if $U \subseteq \mathbb{F}_{p^m}$ satisfies $\mathbb{E}(U) \geq \alpha/2$, and $m = \lceil 2c_p/\alpha \rceil$, then U contains a three-term arithmetic progression.

Let \mathcal{V} denote the sets of all additive subgroups of \mathbb{F}_{p^n} of size p^m . For our proof we will need to establish some facts about \mathcal{V} : First, observe that any sequence of m linearly independent vectors in \mathbb{F}_{p^n} determines a subgroup in \mathcal{V} ; however, each subgroup has many corresponding sequences of m vectors, though each subgroup has the same number of sequences. Now, it is easy to see that the number of sequences of m linearly independent vectors in \mathbb{F}_{p^n} is

$$(p^n - 1)(p^n - p) \cdots (p^n - p^{m-1}) = \epsilon_1 p^{mn}, \text{ where } 1/2 < \epsilon_1 < 1;$$

and, given a subgroup in \mathcal{V} (which can also be thought of as an \mathbb{F}_p vector subspace of dimension m), there are

$$(p^m - 1)(p^m - p) \cdots (p^m - p^{m-1}) = \epsilon_2 p^{m^2}, \text{ where } 1/2 < \epsilon_2 \leq \epsilon_1 < 1,$$

sequences of m linearly independent vectors in \mathbb{F}_{p^n} that span this subgroup. So,

$$|\mathcal{V}| = \epsilon_3 p^{m(n-m)}, \text{ where } 1 \leq \epsilon_3 < 2.$$

Next, suppose that $a \in \mathbb{F}_{p^n}$. We will need to know how many subgroups in \mathcal{V} contain a : Any such subgroup (subspace) can be written as $\text{span}(a) + Z$, where $\dim(Z) = m-1$, and $Z \subseteq \text{span}(a)^\perp$. Thus, Z is any $m-1$ dimensional subspace of an $n-1$ dimensional space; and so, from our bounds on $|\mathcal{V}|$, we deduce that there are $\epsilon_4 p^{(m-1)(n-m)}$, $1/2 < \epsilon_4 < 1$, possibilities for Z , which implies that there are

$$\epsilon_4 p^{(m-1)(n-m)} = \epsilon_5 |\mathcal{V}| p^{m-n}, \text{ where } 1/2 < \epsilon_5 \leq 1,$$

subspaces of \mathbb{F}_{p^n} of dimension m that contain a .

Now, given an arithmetic progression $a, a+d, a+2d$, we note that the progression lies in a coset $b+A$ of an additive subgroup A if and only if

$a \in b + A$ and $d \in A$. Thus, if we define $T'_3(X)$ to be the number of non-trivial three-term arithmetic progressions belonging to a set X , then the sum of the number of non-trivial arithmetic progressions lying in $(b + A) \cap S$, over all $A \in \mathcal{V}$, and $b \in A^\perp$ equals

$$\begin{aligned} \sum_{A \in \mathcal{V}} \sum_{b \in A^\perp} T'_3((b + A) \cap S) &= \sum_{\substack{a, a+d, a+2d \in S \\ d \neq 0}} \sum_{\substack{A \in \mathcal{V} \\ d \in A}} \sum_{\substack{b \in A^\perp \\ a \in b + A}} 1 \\ &= \sum_{\substack{a, a+d, a+2d \in S \\ d \neq 0}} \sum_{\substack{A \in \mathcal{V} \\ d \in A}} 1 \\ &\leq |\mathcal{V}| p^{m-n} T'_3(S). \end{aligned} \quad (23)$$

We now give a lower bound on this first double sum over A and b : We begin with

$$\sum_{A \in \mathcal{V}} \sum_{b \in A^\perp} |(b + A) \cap S| = |\mathcal{V}| |S|, \quad (24)$$

which can be seen by noting that each $s \in S$ lies in exactly one coset $b + A$ of each subgroup $A \in \mathcal{V}$. Now consider all the cosets $b + A$, $A \in \mathcal{V}$, such that

$$|(b + A) \cap S| \geq \alpha |A|/2. \quad (25)$$

We claim that there are more than $|\mathcal{V}| p^{n-m} \alpha/2$ such cosets. To see this, suppose there are fewer than this many cosets. Then, the left-most quantity in (24) is at most

$$(|\mathcal{V}| p^{n-m} \alpha/2) p^m + (|\mathcal{V}| p^{n-m}) (\alpha |A|/2) < |\mathcal{V}| \alpha p^n = |\mathcal{V}| |S|,$$

which would contradict (24).

Thus, there are indeed more than $|\mathcal{V}| p^{n-m} \alpha/2$ cosets satisfying (25). For each such coset $b + A$, since

$$|A| = p^m = p^{\lceil 2c_p/\alpha \rceil},$$

we deduce that $T'_3((b + A) \cap S) \geq 1$; and so,

$$\sum_{A \in \mathcal{V}} \sum_{b \in A^\perp} T'_3((b + A) \cap S) \geq |\mathcal{V}| p^{n-m} \alpha/2.$$

Combining this with (23) we deduce that

$$T'_3(S) \geq p^{2n-2m} \alpha/2 \geq p^{2n} (\alpha/2) \exp(-8\alpha^{-1} c_p \log p).$$

This clearly implies the theorem.

3.4 Resumption of the Proof

From Theorem 5 and (22) we deduce that

$$T_3(V') \geq (\epsilon/4) \exp(-16\epsilon^{-1}c_p \log p) |V|^2.$$

Combining this with (19), we deduce that

$$T_3(g) \leq T_3(f_W) - 2\Delta p^{2n}.$$

This, along with (16) implies

$$\Lambda_3(g) \leq \Lambda_3(f_W) - 2\Delta \leq \Lambda_3(f) - \Delta,$$

which proves the theorem.

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